Safety from holography

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Introduction

Some (realistic and well motivated) theories can be UV well defined only if they have a non-trivial fixed point in the UV

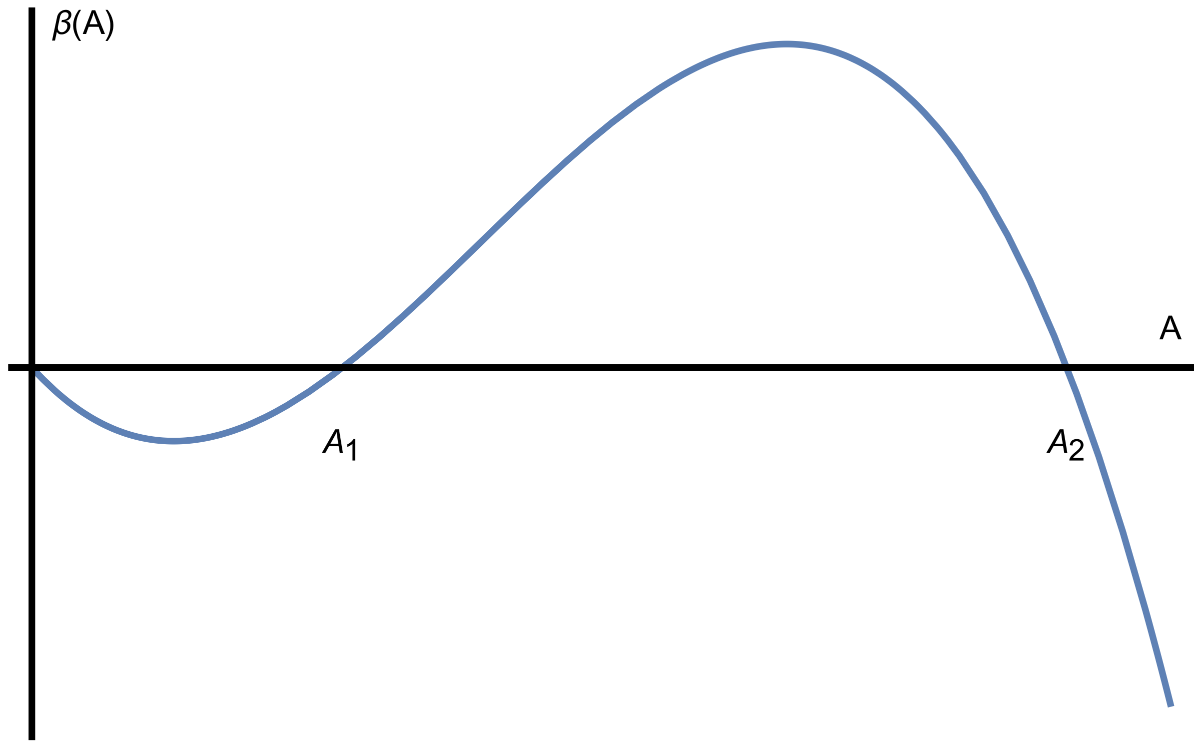
Originally asymptotic safety introduced by Weinberg to give sense to the UV behaviour of gravity: in the UV gravity could have a non-trivial fixed point so the bad divergences we usually encounter is just a consequence we are expanding around a wrong (free) Gaussian fixed point

In this talk we will consider only gauge field theories, no gravity. The simplest example is the gauge-Yukawa-Higgs theory with a proven perturbative (Banks-Zaks) UV fixed point

All three ingredients important in this perturbative case

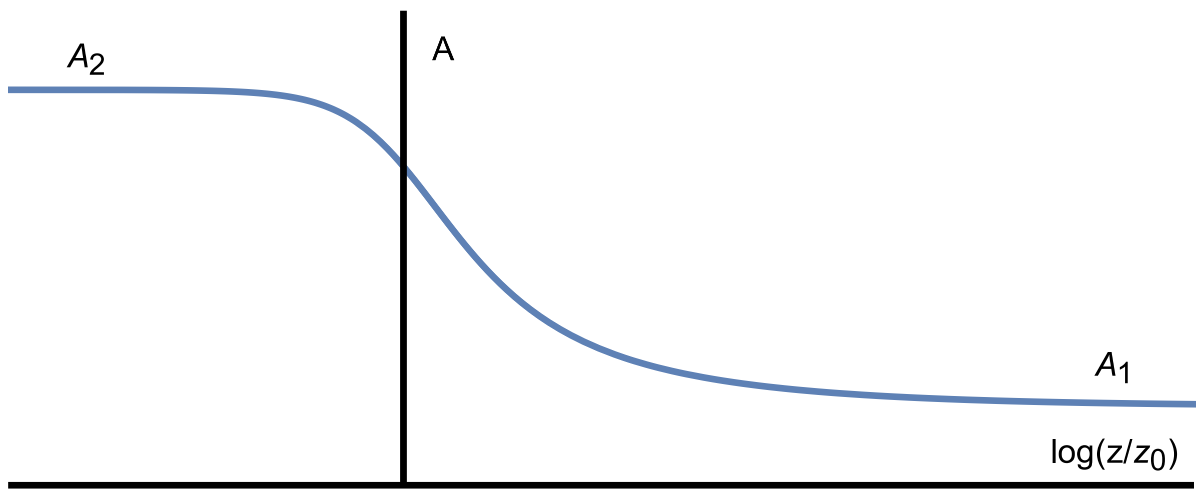
Another simple perturbative example is the same system in $d = 4 - \epsilon$ dimensions

This is the model we would like to reproduce holographically



$$\beta(A) = bA(A - A_1)(A_2 - A) \quad , \quad A = g^2$$

The interested range to study is from $A = A_1$ (IR) to $A = A_2$ (UV)


A_2
A
A_1
$\log(z/z_0)$

To get in field theory the parameters in the right range

$$b > 0 \quad , \quad A_2 > A_1 > 0$$

one needs to consider a separatrix in the phase diagram of the full gauge-Yukawa-Higgs theory

$$\alpha_Y = \alpha_Y(\alpha_g) \quad , \quad \alpha_H = \alpha_H(\alpha_g)$$

Which operator is connected to the gauge coupling?

$$D_\mu = \partial_\mu - igA_\mu$$

By $A_\mu \to \frac{1}{g} A_\mu$

$$D_\mu = \partial_\mu - igA_\mu \to \partial_\mu - iA_\mu$$

(gauge coupling disappears) BUT

$$F^2 \to \frac{1}{g^2} F^2$$

So $\frac{1}{g^2}$ is the coupling in front of $F^2(x)$

We will study the RG running and the behavior of the operator $F^2(x)$ from the holographic point of view: instead of attacking directly the d-dimensional field theory, we will work in a curved $d+1$-dimension al bulk.

- operator : $F^2(x)$
- coupling : A $(= g^2)$

For the AdS/CFT prescription gravity is actually not really important. It can be non-dynamical ($M_{Planck} \to \infty$), its only effect is to make space-time curved

Bulk theory for the dilaton $\chi = \log A$

$$S_{bulk} = \underbrace{\int d^d x}_{\text{our space-time}} \int dz \sqrt{-g} \left(\frac{1}{2} g^{ab} \partial_a \chi \partial_b \chi + V(\chi) \right)$$

Curved space in the bulk (just AdS):

$$ds^2 = \frac{1}{z^2} \left(dz^2 + dx^2 \right)$$

$$UV: \quad z \to 0 \quad , \quad IR: \quad z \to \infty$$

The equation of motion for $\chi(z,x)$

$$\left(z\partial_z\left(z\partial_z - d\right) + z^2\partial_x^2\right)\chi(z,x) = \partial_\chi V(\chi)$$

We solve it by expanding into

- background $\chi(z)$
- perturbation $\xi(z,k)$

$$\chi(z,x) = \chi(z) + \int \frac{d^dk}{(2\pi)^d} e^{ikx}\xi(z,k)$$

AdS/CFT correspondence:

- the RG running of the gauge coupling $g^2(\mu)$ on the boundary corresponds to the profile of the bulk background $A(z) = \exp\left(\chi(z)\right)$
$$z = \frac{1}{\mu}$$
- the behaviour of the operator $F^2(x)$ on the boundary corresponds to the limiting $z \to 0$ behaviour of the bulk perturbation $\xi(z, x)$

We will see it more precisely soon

The background

The first part, the RG running, i.e. the z-profile of the background field $A(z)$ is easy:

$$\mu\partial_\mu A = \beta(A)$$

transforms into

$$-z\partial_z A = bA(A - A_1)(A_2 - A)$$

This eq. is the BPS equation from a potential

$$V(A) = \frac{1}{2}\left(A\partial_A W\right)^2 - dW$$

where the "superpotential" W is defined through

$$\beta(A) = -A^2\partial_A W$$

In other words, a properly chosen W gives an equation of motion which solution is equal to the solution of the RGE above:

$$z(A) = \exp\left(-\int \frac{dA}{\beta(A)}\right)$$

In our concrete case

$$\left(\frac{z}{z_0}\right)^{bA_1A_2(A_2-A_1)} = A^{A_2-A_1}\frac{(A_2-A)^{A_1}}{(A-A_1)^{A_2}}$$

Important:

$$
\begin{aligned}
z \to \infty \text{ (IR)} \quad &: \quad (A - A_1) \to c_1\, z^{d-\Delta_1} \\
z \to 0 \text{ (UV)} \quad &: \quad (A_2 - A) \to c_2\, z^{d-\Delta_2}
\end{aligned}
$$

From the boundary field theory point of view

$$
\Delta_{F^2} = d + \partial_A \beta - \frac{\beta}{A}
$$

one gets

$$
\begin{aligned}
\Delta_1 \quad &= \quad d + bA_1(A_2 - A_1) \\
\Delta_2 \quad &= \quad d - bA_2(A_2 - A_1)
\end{aligned}
$$

are the IR and UV dimensions of the operator $F^2(x)$.

Always:

$$\begin{aligned} \Delta_1 &\geq d \\ \Delta_2 &\leq d \end{aligned}$$

This is true in any model of this type:

IR dimension never smaller than UV dimension

This follows essentially from the a-theorem

The perturbation

What about the operator $F^2(x)$? Quantities which are interesting are the correlators

$$\langle F^2(x_1)F^2(x_2)\dots F^2(x_n)\rangle$$

We will limit ourselves to the simplest, the propagator

$$G(k) = \int d^dx\, e^{-ikx}\langle F^2(x)F^2(0)\rangle$$

We expect (dimensional analysis) them to behave like

$$k \to 0 \text{ (IR)} \quad : \quad G \to k^{2\Delta_1 - d}$$
$$k \to \infty \text{ (UV)} \quad : \quad G \to k^{2\Delta_2 - d}$$

This is from the holographic point of view the operator dimensions $\Delta_{1,2}$ as functions of model parameters b, $A_{1,2}$ which obviously agree with the boundary field theory computation

We will check Δ_1 later on

Linearized eq. for perturbations:

$$z\partial_z \left(z\partial_z - d\right) \xi(z,k) = \left(V''(\chi(z)) + (kz)^2\right) \xi(z,k)$$

A solution (well-behaved at $z \to \infty$) for $\xi(z,k)$ gives all you need for the propagator. The AdS/CFT prescription is

$$z \to 0 \quad : \quad \xi(z,k) \to \xi_0(k) \left(z^{d-\Delta_2} + G(k) z^{\Delta_2}\right)$$

In fact there are 2 independent solutions of the second order linear differential equation, one starting with z^{Δ_2}, the other with $z^{d-\Delta_2}$ for $z \to 0$

Unfortunately we do not know such a solution.

- numerical work (not here)
- approximate V with some form with known analytic solution, for example a piece-wise quadratic form. Then everything analytic (not here)
- for the IR behaviour the so-called matching method is enough

Idea:

solve approximately the equation in two different regimes

Then match the two approximate solutions where both approximations are valid

- small k, any z:

$$\xi_{k\to 0}(z,k) = c_1(k)\xi_1(z) + c_2(k)\xi_2(z)$$

where

$$\begin{aligned} \xi_1(z) &= z\partial_z\chi(z) \\ \xi_2(z) &= \xi_1(z)\int^z dy\,\frac{y^{d-1}}{\xi_1^2(y)} \end{aligned}$$

are exact solutions of the $k=0$ equations

$$z^2\partial_z^2\xi_{1,2}(z) - (d-1)z\partial_z\xi_{1,2}(z) = V''(\chi(z))\xi_{1,2}(z)$$

Explicitly for our case

$$\xi_1(z) \sim (A(z) - A_1)(A_2 - A(z))$$

$$\xi_2(z) \sim (A(z) - A_1)(A_2 - A(z))^{\alpha_2+2}$$

$$\times F_1\left(\alpha_2+1; -\alpha_1, -\alpha_3; \alpha_2+2; \frac{A_2 - A(z)}{A_2 - A_1}, \frac{A_2 - A(z)}{A_2}\right)$$

with

$$\alpha_1 = -\frac{d}{bA_1(A_2 - A_1)} - 3 \quad \left(= \frac{d}{d - \Delta_1} - 3\right)$$

$$\alpha_2 = \frac{d}{bA_2(A_2 - A_1)} - 3 \quad \left(= \frac{d}{d - \Delta_2} - 3\right)$$

$$\alpha_3 = \frac{d}{bA_1 A_2} - 1 \quad \left(= -\frac{d}{d - \Delta_1} - \frac{d}{d - \Delta_2} - 1\right)$$

Appell's series

$$F_1(a;b,c;d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}\, m!\, n!} x^m y^n$$

converges for $|x|, |y| < 1$

Pochhammer symbol

$$(q)_k = \frac{\Gamma(q+k)}{\Gamma(q)}$$

- Large z, any k:

$$\xi_{z\to\infty}(z,k) = \frac{(kz)^2}{2} K_{\Delta_1-d/2}(kz)$$

  where we discarded the bad $I_{\Delta_1-d/2}(kz)$
  ($\to \exp(kz)$ at large kz)

The matching can be done at small kz, where both approximations are valid

The second one gets expanded for $kz \ll 1$ like

$$\begin{aligned}
\xi_\infty(z,k) &\approx \left[\Gamma\left(\frac{d}{2}-\Delta_1\right)\left(\frac{kz}{2}\right)^{\Delta_1}+\ldots\right] \\
&+ \left[\Gamma\left(-\frac{d}{2}+\Delta_1\right)\left(\frac{kz}{2}\right)^{d-\Delta_1}+\ldots\right]
\end{aligned}$$

while for the first one ($\xi_{k\to 0}$) it is a bit more tricky

Schematically it goes like that: for $z \to \infty$ (IR)

$$\xi_1(z) \approx a_\infty(k) \left(\frac{z}{z_0}\right)^{\Delta_1} + \tilde{a}_\infty(k) \left(\frac{z}{z_0}\right)^{d-\Delta_1}$$

$$\xi_2(z) \approx b_\infty(k) \left(\frac{z}{z_0}\right)^{\Delta_1} + \tilde{b}_\infty(k) \left(\frac{z}{z_0}\right)^{d-\Delta_1}$$

with calculable a_∞, $\tilde{a}_\infty$, b_∞, $\tilde{b}_\infty$

In this way for large z, small k, but fixed $kz \ll 1$ we get

$$\xi_{k\to 0}(z,k) \approx \xi_{z\to\infty}(z,k)$$

i.e.

$$c_1(k)a_\infty + c_2(k)\tilde{a}_\infty = \Gamma\left(\frac{d}{2} - \Delta_1\right)\left(\frac{kz_0}{2}\right)^{\Delta_1}$$

$$c_1(k)b_\infty + c_2(k)\tilde{b}_\infty = \Gamma\left(-\frac{d}{2} + \Delta_1\right)\left(\frac{kz_0}{2}\right)^{d-\Delta_1}$$

It turns out that

$$\frac{c_1(k)}{c_2(k)} = const \times (kz_0)^{2\Delta_1 - d} + \ldots$$

Then look into the opposite limit, $z \to 0$ (UV):

$$\begin{aligned} \xi_1(z) &\approx a_0 \left(\frac{z}{z_0}\right)^{\Delta_2} + \tilde{a}_0 \left(\frac{z}{z_0}\right)^{d-\Delta_2} \\ \xi_2(z) &\approx b_0 \left(\frac{z}{z_0}\right)^{\Delta_2} + \tilde{b}_0 \left(\frac{z}{z_0}\right)^{d-\Delta_2} \end{aligned}$$

and from the definition of the F^2 propagator G

$$\xi(z,k) \to \xi_0(k) \left(z^{d-\Delta_2} + G(k) z^{\Delta_2}\right)$$

it follows the propagator

$$G(k) = \frac{a_0 c_1(k) + b_0 c_2(k)}{\tilde{a}_0 c_1(k) + \tilde{b}_0 c_2(k)}$$

This was general, in our case for $kz_0 \ll 1$ (IR)

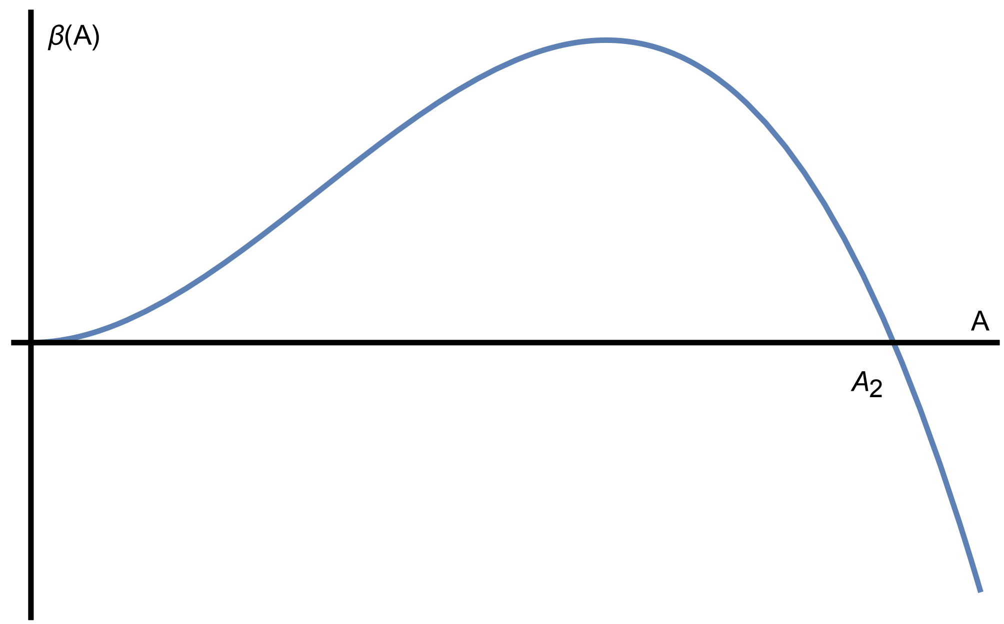
$$G(k) \sim \frac{c_1(k)}{c_2(k)} \sim k^{2\Delta_1 - d}$$

as we expected

Other possibilities

One could study the IR free theory, i.e. with a double zero of $\beta(A)$ at $A = 0$, i.e. for $A_1 \to 0$

$$\beta(A) = bA^2(A_2 - A)$$



The computation much more involved because of the appearance of multiple logs.

This would be the case for large N_f UV fixed points:

In the IR the dimension

$$\Delta_{F^2}|_{IR} = d$$

as for a Gaussian fixed point.

In the UV the formula for Δ_{F^2} gives a very negative quantity, which is at odds with unitarity:

$$\Delta_{\mathcal{O}} \geq 1$$

for any scalar operator $\mathcal{O}$

The standard interpretation in this case is that once $\mathcal{O}$'s dimension has reached 1 from above, the operator $\mathcal{O}$ is a free operator, not coupled to anybody, and so remains such, so

$$\Delta_{F^2}|_{UV} = 1$$

The main problem with such a set-up is the matching

$$\begin{aligned}\xi_{z\to\infty}(z,k) &= \frac{(kz)^2}{2}K_{\Delta_1-d/2}(kz) \\ &= \frac{(kz)^2}{2}K_{d/2}(kz) \\ &\to 1-\frac{1}{16}(kz)^4\log(kz)+\ldots\end{aligned}$$

Notice a single log

On the other side the leading terms for $k \to 0$ at large z (IR)

$$\begin{aligned} \xi_1(z) &\sim const + \ldots \\ \xi_2(z) &\sim z^4 (\log z)^2 + \ldots \end{aligned}$$

There are double logs

Naive matching does not work

Work in progress

Conclusions

- Asymptotic safety may give sense to theories with an apparent pathology (Landau pole); it gives and alternative to asymptotically free theories
- the holographic version of a perturbative UV safe theory in $d = 4 - \epsilon$ dimensions explicitly reproduced here
- the hologram of the $d = 4$ (for ex. large N_f non-perturbative) UV fixed point could be obtained as well providing the double zeros and consequently the nasty logs will be tamed